Particle Motion Problems

Particle motion problems deal with particles that are moving along the $x$– or $y$– axis. Thus, we are speaking of horizontal or vertical movement. The position, velocity, or acceleration of a particle’s motion are DEFINED by functions, but the particle DOES NOT move along the graph of the function. It moves along an axis. Most of the time, we speak of movement along the $x$– axis. In units 6 and 7, particle motion is revisited. At that time, we will deal more with vertical motion. For the time, we will focus on horizontal motion of particles.

In this lesson, we develop the ideas of velocity and acceleration in terms of position. We will speak of two types of velocities and accelerations. Let’s define average and instantaneous velocity in the box below.

### Average and Instantaneous Velocity

A particle’s position is given by the function $p(t) = e^t \sin t$, where $p(t)$ is measured in centimeters and $t$ is measured in seconds. Answer the following questions.

What is the average velocity on the interval $t = 1$ to $t = 3$ seconds? Indicate appropriate units of measure.

What is the instantaneous velocity of the particle at time $t = 1.5$. Indicate appropriate units of measure.
Before we proceed, a connection needs to be made. When given a function, \( f(x) \), how did we find the slope of the secant line on the interval from \( x = a \) to \( x = b \)? In terms of position of a particle, to what does the slope of the secant line correspond? To what does the instantaneous velocity correspond?

**Average and Instantaneous Acceleration**

A particle’s position is given by the function \( p(t) = e^t \sin t \), where \( p(t) \) is measured in centimeters and \( t \) is measured in seconds. Answer the following questions.

What is the average acceleration on the interval \( t = 1 \) to \( t = 3 \) seconds? Indicate appropriate units of measure.

What is the instantaneous acceleration of the particle at time \( t = 1.5 \).
In summary, let’s correlate the concepts of position, velocity, and acceleration to what we already know about a function and its first and second derivative.

Let’s summarize our relationships between position, velocity and acceleration below.

<table>
<thead>
<tr>
<th>Velocity</th>
<th>Position (Motion of the Particle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is = 0 or is undefined</td>
<td></td>
</tr>
<tr>
<td>Is &gt; 0</td>
<td></td>
</tr>
<tr>
<td>Is &lt; 0</td>
<td></td>
</tr>
<tr>
<td>Changes from positive to negative</td>
<td></td>
</tr>
<tr>
<td>Changes from negative to positive</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Acceleration</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is = 0 or is undefined</td>
<td></td>
</tr>
<tr>
<td>Is &gt; 0</td>
<td></td>
</tr>
<tr>
<td>Is &lt; 0</td>
<td></td>
</tr>
<tr>
<td>Changes from positive to negative</td>
<td></td>
</tr>
<tr>
<td>Changes from negative to positive</td>
<td></td>
</tr>
</tbody>
</table>
The graph below represents the position, $s(t)$, of a particle which is moving along the $x$ axis.

- At which point(s) is the velocity equal to zero? Justify your answer.

- At which point(s) does the acceleration equal zero? Justify your answer.

- On what interval(s) is the particle’s velocity positive? Justify your answer.

- On what interval(s) is the particle’s velocity negative? Justify your answer.

- On what interval(s) is the particle’s acceleration positive? Justify your answer.

- On what interval(s) is the particle’s acceleration negative? Justify your answer.
Five Commandments of Particle Motion

1. 

2. 

3. 

4. 

5. 

Suppose the velocity of a particle is given by the function \( v(t) = (t + 2)(t + 4)^2 \) for \( t \geq 0 \), where \( t \) is measured in minutes and \( v(t) \) is measured in inches per minute. Answer the questions that follow.

a. Find the values of \( v(3) \) and \( v'(3) \). Based on these values, describe the speed of the particle at \( t = 3 \).

b. On what interval(s) is the particle moving to the left? Right? Show your analysis and justify your answer.
The graph of the velocity \( v(t) \), in feet per second, of a car traveling on a straight road, for \( 0 \leq t \leq 50 \) is shown below. A table of values for \( v(t) \), at 5 second intervals of time, is also shown to the right of the graph.

<table>
<thead>
<tr>
<th>( t ) (seconds)</th>
<th>( v(t) ) (feet per second)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>55</td>
</tr>
<tr>
<td>25</td>
<td>70</td>
</tr>
<tr>
<td>30</td>
<td>78</td>
</tr>
<tr>
<td>35</td>
<td>81</td>
</tr>
<tr>
<td>40</td>
<td>75</td>
</tr>
<tr>
<td>45</td>
<td>60</td>
</tr>
<tr>
<td>50</td>
<td>72</td>
</tr>
</tbody>
</table>

a. During what interval(s) of time is the acceleration of the car positive? Give a reason for your answer.

b. Find the average acceleration of the car over the interval \( 0 \leq t \leq 50 \). Indicate units of measure.

c. Find one approximation for the acceleration of the car at \( t = 40 \). Show the computations you used to arrive at your answer. Indicate units of measure.

d. Is the speed of the car increasing or decreasing at \( t = 40 \)? Give a reason for your answer.
Two runners, $A$ and $B$, run on a straight racetrack for $0 \leq t \leq 10$ seconds. The graph below, which consists of two line segments, shows the velocity, in meters per second, of Runner $A$. The velocity, in meters per second, of Runner $B$ is given by the function $v$ defined by $v(t) = \frac{24t}{2t+3}$.

![Graph showing velocity vs time for Runner A.](image)

a. Find the velocity of Runner $A$ and the velocity of Runner $B$ at $t = 2$ seconds. Indicate units of measure.

b. Find the acceleration of Runner $A$ and the acceleration of Runner $B$ at time $t = 2$ seconds. Indicate units of measure.
2002 AP Calculus AB #3 (Partial)

An object moves along the $x$– axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by the function $v(t) = \sin\left(\frac{\pi}{3} t\right)$.

a. What is the acceleration of the object at time $t = 4$?

b. Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.

Statement II: For $3 < t < 4.5$, the speed of the object is decreasing.

Are either or both of these statements correct? For each statement, provide a reason why it is correct or not correct.
A particle moves along the \( x \)–axis so that its velocity at time \( t \) is given by 
\[
v(t) = -(t + 1) \sin \left( \frac{t^2}{2} \right).
\]

a. Find the acceleration of the particle at \( t = 2 \). Is the speed of the particle increasing at \( t = 2 \)? Explain why or why not.

b. Find all times in the open interval \( 0 < t < 3 \) when the particle changes direction. Justify your answer.
More on Particle Motion

**Finding Net and Total Distance**

The graph below represents the velocity, \( v(t) \) which is measured in meters per second, of a particle moving along the \( x \)– axis.

![Graph of velocity vs. time]

At what value(s) of \( t \) does the particle have no acceleration on the interval (0, 10)? Justify your answer.

Express the acceleration, \( a(t) \), as a piecewise-defined function on the interval (0, 10).

For what value(s) of \( t \) is the particle moving to the right? To the left? Justify your answer.

Find the average acceleration of the particle on the interval [1, 8]. Show your work.
Definition of Net Distance:

Definition of Total Distance:

If a particle is moving in the same direction the entire amount of time, what can be said about the net distance and the total distance?

To Find the Net Distance a Particle Travels on an Interval

To Find the Total Distance a Particle Travels on an Interval

The position of a particle is given by the function \( p(t) = 2t^3 - 6t^2 + 8t \) where \( p(t) \) is measures in centimeters. Find the net and total distance the particle travels from \( t = 1.5 \) seconds to \( t = 4 \) seconds.
The position of a particle is given by the function \( p(t) = e^{2t} - 8t \) where \( p(t) \) is measures in feet. Find the net and total distance the particle travels from \( t = 0.5 \) minutes to \( t = 1.5 \) minutes.

The position of a particle is given by the function \( p(t) = t + 2 \sin t \) where \( p(t) \) is measures in feet. Find the net and total distance the particle travels from \( t = \frac{\pi}{6} \) minutes to \( t = \frac{5\pi}{4} \) minutes.
NO CALCULATOR PERMITTED

A particle moves along the x-axis so that its position at any time \( t \geq 0 \) is given by the function 
\[ p(t) = t^3 - 4t^2 - 3t + 1, \]
where \( p \) is measured in feet and \( t \) is measured in seconds.

a. Find the average velocity on the interval \( t = 1 \) and \( t = 2 \) seconds. Give your answer using correct units.

b. On what interval(s) of time is the particle moving to the left? Justify your answer.

c. Using appropriate units, find the value of \( p'(3) \) and \( p''(3) \). Based on these values, describe the motion of the particle at \( t = 3 \) seconds. Give a reason for your answer.

d. What is the maximum velocity on the interval from \( t = 1 \) to \( t = 3 \) seconds. Show the analysis that leads to your conclusion.

e. Find the total distance that the particle moves on the interval \([1, 5]\). Show and explain your analysis.
A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time $t$ minutes, where $v$ is a differentiable function of $t$. Selected values of $v(t)$ for $0 \leq t \leq 40$ are shown in the table below.

<table>
<thead>
<tr>
<th>$t$ (min)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$ (miles per min)</td>
<td>7.0</td>
<td>9.2</td>
<td>9.5</td>
<td>7.0</td>
<td>4.5</td>
<td>2.4</td>
<td>2.4</td>
<td>4.3</td>
<td>7.3</td>
</tr>
</tbody>
</table>

a. Find the average acceleration on the interval $5 \leq t \leq 20$. Express your answer using correct units of measure.

b. Based on the values in the table, on what interval(s) is the acceleration of the plane guaranteed to equal zero on the open interval $0 < t < 40$? Justify your answer.

c. Does the data represent velocity values of the plane moving away from its point of origin or returning to its point of origin? Give a reason for your answer.

d. The function $f$, defined by $f(t) = 6\cos\left(\frac{t}{10}\right)+3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \leq t \leq 40$. According to this model, what is the acceleration of the plane at $t = 23$? What does this value indicate about the velocity at $t = 23$? Justify your answer, indicating units of measure.
AP Free Response and Multiple Choice Practice

NO CALCULATOR

A particle moves along the x–axis with velocity at time \( t \geq 0 \) given by \( v(t) = -1 - e^{1-t} \).

a. Find the acceleration of the particle at \( t = 3 \).

b. Is the speed of the particle increasing at \( t = 3 \)? Give a reason for your answer.

c. Find all values of \( t \) at which the particle changes direction. Justify your answer.

d. The function \( p(t) = e^{1-t} - t \) models the position of the particle for \( t \geq 0 \). Find the total distance that the particle traveled on the time interval \( 0 \leq t \leq 3 \).
A car is traveling on a straight road. For $0 \leq t \leq 24$ seconds, the car’s velocity, $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph below.

![Graph showing velocity over time](image)

(a) For what interval(s) of time does the car have zero acceleration? Show the work and explain the analysis that leads to your answer.

(b) For each value of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car’s acceleration at time $t$ in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of $v$ over the interval $8 < t < 20$. Does the Mean Value Theorem guarantee a value of $c$, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?
You must show your work to earn credit for the following. You will need to use a calculator for these.

If \( f(x) = \sin\left(\frac{x}{2}\right) \), then there exists a number \( c \) on the interval \( \frac{\pi}{2} < x < \frac{3\pi}{2} \) that satisfies the conclusion of the Mean Value Theorem. Which of the following values could be \( c \)?

(A) \( \frac{2\pi}{3} \)  
(B) \( \frac{3\pi}{4} \)  
(C) \( \frac{5\pi}{6} \)  
(D) \( \pi \)  
(E) \( \frac{3\pi}{2} \)

A particle moves along a line so that at time \( t \), where \( 0 \leq t \leq \pi \), its position is given by

\[
s(t) = -4\cos t - \frac{t^2}{2} + 10.
\]

What is the velocity of the particle when its acceleration is zero?

(A) \(-5.19\)  
(B) \(0.74\)  
(C) \(1.32\)  
(D) \(2.55\)  
(E) \(8.13\)
The graph of the function \( y = x^3 + 6x^2 + 7x - 2 \cos x \) changes concavity at \( x = \)

(A) –1.58  \hspace{1cm} (B) –1.63  \hspace{1cm} (C) –1.67  \hspace{1cm} (D) –1.89  \hspace{1cm} (E) –2.33

If \( y = 2x - 8 \), what is the minimum value of the product of \( xy \)?

(A) –16  \hspace{1cm} (B) –8  \hspace{1cm} (C) –4  \hspace{1cm} (D) 0  \hspace{1cm} (E) 2
Solving Optimization Problems

General Approach to Solving Optimization Problems

1. 

2. 

3. 

4. 

5. 

Example 1

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume? What is the maximum volume?
Example 2

A box is to be built from a rectangular piece of cardboard that is 25 cm wide and 40 cm long by cutting out a square from each corner and then bending up the sides. Find the size of the corner square with will produce a container that will hold the most amount of soup.

Example 3

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1 ½ inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

Example 4

A rectangle is bounded by the $x$ and $y$ axes and the graph of $y = 3 - \frac{1}{2}x$. What length and width should the rectangle have so that its area is a maximum?
Example 5

A solid is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the solid is 12 cubic centimeters. Find the radius of the cylinder that produces the minimum surface area.

Example 6

The profit $P$ (in thousands of dollars) for a company spending an amount of $s$ (in thousands of dollars) on advertising is $P = -\frac{1}{10}s^3 + 6s^2 + 400$. Find the amount of money the company should spend on advertising in order to yield a maximum profit.
Example 7

Determine the point on the line \( y = 2x + 3 \) so that the distance between the line and the point \( (1, 2) \) is a minimum.

Example 8

A rectangle ABCD with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of \( y = -4x^2 + 4 \) as shown in the figure below. Find the \( x \) and \( y \) coordinates of the point C so that the area of the rectangle is a maximum.